ON THE DISCRETE SPECTRUM OF ROBIN LAPLACIANS IN CONICAL DOMAINS

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ABSTRACT. We discuss several geometric conditions guaranteeing the finiteness or the infiniteness of the discrete spectrum for Robin Laplacians on conical domains.

1. INTRODUCTION

Let $\Omega \subset \mathbb{R}^\nu$ be an open set with a sufficiently regular boundary. For $\alpha > 0$, denote by Q_α^Ω the self-adjoint operator in $L^2(\Omega)$ acting as the Laplacian $u \mapsto -\Delta u$ in Ω with the Robin boundary condition $\partial u/\partial n = \alpha u$ on $\partial\Omega$, where $\alpha > 0$ is a fixed constant and $\partial/\partial n$ means the derivative in the direction of the outer unit normal n. The above definition should be understood in a suitable weak sense: Q_α^Ω is the self-adjoint operator in $L^2(\Omega)$ generated by the quadratic form

$$q_\alpha^\Omega(u, u) = \int_\Omega |\nabla u|^2 \mathrm{d}x - \alpha \int_{\partial\Omega} u^2 \mathrm{d}\sigma, \quad \mathcal{D}(q_\alpha^\Omega) = H^1(\Omega),$$

where σ stands for the $(\nu - 1)$-dimensional Hausdorff measure on $\partial\Omega$. The study of the spectral properties of Q_α^Ω arises in numerous applications. For example, the paper [14] shows a link between the eigenvalue problem and the long-time dynamics related to some reaction-diffusion process. The paper [6] discusses the stochastic meaning of the eigenvalues. Various properties of Q_α^Ω appear to be of importance for problems of surface superconductivity, see [2, 10]. In the present contribution, we are going to discuss some spectral properties of Q_α^Ω for a special class of non-compact domains Ω.

By a *cone* we mean a connected Lipschitz domain $\Lambda \subset \mathbb{R}^\nu$, $\nu \geq 2$, for which there exists a point $V \in \mathbb{R}^\nu$ (vertex of the cone) such that for any $x \in \mathbb{R}^\nu$ the condition $V + x \in \Lambda$ implies $V + tx \in \Lambda$ for all $t > 0$. A cone is called *smooth* if its boundary is non-empty and is C^2 with the possible exception of a vertex. Clearly, for $\nu \geq 3$ a cone Λ with a vertex V is smooth iff its *cross section*

$$\Sigma_\Lambda := \{y \in \mathbb{R}^\nu, |y| = 1 : V + y \in \Lambda\} \subset \mathbb{S}^{\nu-1}$$

is a domain of $\mathbb{S}^{\nu-1}$ with a non-empty C^2 boundary, while in $\mathbb{R}^2$ any cone is smooth. A Lipschitz domain Ω is called a *conical* one if there exists a cone Λ such that the Ω coincides with Λ outside a ball, and the associated cone $\Lambda \equiv \Lambda(\Omega)$ is then uniquely defined. A conical domain is said to be *smooth at infinity* if the associated cone is smooth.

The Robin Laplacians on cones play a special role in the study of the bottom of the spectrum $E_1^\Omega(\alpha)$ of Q_α^Ω as α becomes large. As shown by Levitin and Parnovski [15], for piecewise smooth Ω one has

$$E_1^\Omega(\alpha) = -C_\Omega \alpha^2 + o(\alpha^2) \text{ as } \alpha \text{ tends to } +\infty \tag{1}$$

1

with $C_\Omega = -\inf_{x \in \partial\Omega} E_1^{\Lambda_x}(1)$, where Λ_x is the tangent cone at $\partial\Omega$ at the point x. The paper [15] also presented several cases for which the quantity $E_1^{\Lambda_x}(1)$ can be calculated explicitly using some constructions of the spherical geometry. We remark that there are various refinements of the asymptotics (1), see e.g. [4, 7, 8, 11, 12, 16–20].

On the other hand, conical geometries attracted an attention in the study of Laplace-type operators due to the fact that they may produce infinitely many discrete eigenvalues. It seems that such an effect was found for the first time by Exner and Tater [9] who showed that the Dirichlet Laplacian in a rotationally symmetric conical layer in three dimensions has an infinite discrete spectrum. Recently, Dauge, Ourmières-Bonafos and Raymond [5] provided a more detailed study by calculating the accumulation rate of the eigenvalues. Behrndt, Exner and Lotoreichik [1] showed the infiniteness of the discrete spectrum for the Schrödinger operators with δ-interaction on rotationally symmetric conical surfaces or their compactly supported perturbations. We remark that all these works used the invariance of the domain with respect to the rotations.

The principal aim of the present note is to discuss the cardinality of the discrete spectrum for Robin Laplacians in conical domains smooth at infinity. In particular, we will show that the infiniteness of the discrete spectrum can be guaranteed by a rather simple sufficient condition: the strict positivity of the mean curvature of the boundary on a half-line originating from a vertex, see Theorem 6. Furthermore, in three dimensions this condition is (in a sense) necessary as well, see Corollary 10. Our constructions are based on a reduction of the analysis of conical domains to the case of strongly coupled Robin Laplacians carried out by Pankrashkin and Popoff [20].

Recall that the principal curvatures of a $(\nu - 1)$-dimensional smooth submanifold S in $\mathbb{R}^\nu$ at a point s are defined as the eigenvalues of the shape operator $dn(s)$, where $s \mapsto n(s)$ is a smooth unit normal and d stands for the differential. The mean curvature $H(s)$ is then the arithmetic mean of the principal curvatures, i.e. $H(s) := \frac{1}{\nu-1} \operatorname{tr} dn(s)$. The definition depends on the orientation (the change of the orientation implies the sign change for the curvatures), but in all the cases below we deal with surfaces which are boundaries of open sets, and the mean curvature will always be calculated with respect to the *outer* unit normal. In particular, the principal curvatures and the mean curvature of the boundary of a convex domain are non-negative.

Let us also recall the min-max principle for the eigenvalues. Let Q be a self-adjoint operator semibounded from below and q be its quadratic form. Let $N \in \mathbb{N}$ and

$$E_N(Q) := \inf_{\substack{L \subset \mathcal{D}(q) \\ \dim L = N}} \sup_{u \in L, \, u \neq 0} \frac{q(u,u)}{\|u\|^2},$$

then

- either $E_N(Q) < \inf \operatorname{spec}_{\mathrm{ess}} Q$ and $E_N(Q)$ is the Nth eigenvalue of Q when numbered in the non-decreasing order and counted with multiplicities,
- or $E_k(Q) = \inf \operatorname{spec}_{\mathrm{ess}} Q$ for all $k \geq N$.

2. Essential spectrum

In order to be able to use the min-max principle we give first a quite standard result on the essential spectrum.

Theorem 1. *Let $\nu \geq 2$ and $\Omega \subset \mathbb{R}^\nu$ be a conical domain smooth at infinity, then* $\operatorname{spec}_{\mathrm{ess}} Q_\alpha^\Omega = [-\alpha^2, +\infty)$ *for any $\alpha > 0$.*

Proof. An easy adaptation of Persson's theorem [13, Section 14.4] gives

$$\inf \operatorname{spec}_{\mathrm{ess}} Q_\alpha^\Omega = \sup_K \inf \left\{ q_\alpha^\Omega(u, u) : u \in C_c^\infty(\mathbb{R}^\nu \setminus K), \|u\|_{L^2(\Omega)} = 1 \right\},$$

where the supremum is over all compact subsets $K \subset \mathbb{R}^\nu$. Consider the cone $\Lambda := \Lambda(\Omega)$ and, furthermore, let Θ be a C^2 *smooth* domain coinciding with Ω (and, hence, with Λ) outside a ball. The above characterization of the bottom of the essential spectrum implies

$$\inf \operatorname{spec}_{\mathrm{ess}} Q_\alpha^\Omega = \inf \operatorname{spec}_{\mathrm{ess}} Q_\alpha^\Theta = \inf \operatorname{spec}_{\mathrm{ess}} Q_\alpha^\Lambda \tag{2}$$

for any α. We may use some results of [20] to study Q_α^Θ. In particular, see [20, Section 7], for $\alpha \to +\infty$ one has

$$\inf \operatorname{spec}_{\mathrm{ess}} Q_\alpha^\Theta \geq -\alpha^2 - (\nu - 1)H_\infty \alpha + o(\alpha), \quad H_\infty := \limsup_{s \to \infty, \, s \in \partial\Omega} H(s),$$

where H is the mean curvature. In our case one has clearly $H_\infty = 0$, hence, $\inf \operatorname{spec}_{\mathrm{ess}} Q_\alpha^\Lambda = \inf \operatorname{spec}_{\mathrm{ess}} Q_\alpha^\Theta \geq -\alpha^2 + o(\alpha)$. On the other hand, using the invariance of Λ with respect to the dilations we obtain

$$\inf \operatorname{spec}_{\mathrm{ess}} Q_\alpha^\Lambda = \alpha^2 \inf \operatorname{spec}_{\mathrm{ess}} Q_1^\Lambda,$$

which gives $\inf \operatorname{spec}_{\mathrm{ess}} Q_1^\Lambda \geq -1$ and then $\inf \operatorname{spec}_{\mathrm{ess}} Q_\alpha^\Lambda \geq -\alpha^2$ for any $\alpha > 0$. Finally, using (2),

$$\operatorname{spec}_{\mathrm{ess}} Q_\alpha^\Omega \subset [-\alpha^2, +\infty) \text{ for any } \alpha > 0. \tag{3}$$

Now we are going to show the inclusion

$$[-\alpha^2, +\infty) \subset \operatorname{spec} Q_\alpha^\Omega. \tag{4}$$

This can be easily done by constructing approximate eigenfunctions. Namely, let $\phi \in C^\infty(\mathbb{R})$ with $\phi(x) = 0$ for $x \leq 0$ and $\phi(x) = 1$ for $x \geq 1$. For $N \in \mathbb{N}$, set

$$\psi_N(x) := \phi(x - N)\phi(2N - x), \quad \varphi_N(x) := \phi(\sqrt{N} - x).$$

Let V be a vertex of the cone Λ. Denote by $(r, \theta) \in \mathbb{R}_+ \times \mathbb{S}^{\nu-1}$ the spherical coordinates centered at V. Furthermore, for $\theta \in \mathbb{S}^{\nu-1}$, let $d(\theta)$ denote the geodesic distance in $\mathbb{S}^{\nu-1}$ between θ and the boundary $\partial\Sigma$ of the cross section $\Sigma = \Sigma_\Lambda$. Take an arbitrary $k > 0$ and consider the functions u_N given in the spherical coordinates by

$$u_N(r, \theta) = \psi_N(r) \sin(kr) e^{-\alpha r d(\theta)} \varphi_N\big(r d(\theta)\big).$$

One easily checks that for large N one has $u_N \in \mathcal{D}(Q_\alpha^\Omega)$ with $\|u_N\|_{L^2(\Omega)} \geq c N^{\frac{\nu-1}{2}}$, where $c > 0$ is independent of N. Recall that the Laplacian in the spherical coordinates takes the form

$$-\Delta \simeq -\frac{\partial^2}{\partial r^2} - \frac{\nu - 1}{r} \frac{\partial}{\partial r} - \frac{1}{r^2} \Delta_{\mathbb{S}^{\nu-1}},$$

where $\Delta_{\mathbb{S}^{\nu-1}}$ is the Laplace-Beltrami operator on $\mathbb{S}^{\nu-1}$. One easily checks that for the functions f of the form $f(\theta) := b\big(d(\theta)\big)$ we have, if $d(\theta) < \varepsilon$ and ε is small,

$$\Delta_{\mathbb{S}^{\nu-1}} f(\theta) = \big(1 + \mathcal{O}(\varepsilon)\big) b''\big(d(\theta)\big) + \mathcal{O}(\varepsilon) b'\big(d(\theta)\big).$$

In particular, for large N we have

$$\Delta_{\mathbb{S}^{\nu-1}} u_N(r,\theta) = \psi_N(r)\sin(kr)r^2 e^{-\alpha r d(\theta)}$$

$$\times \left\{ \left(1 + \mathcal{O}(N^{-1/2})\right) \cdot \left[\alpha^2\varphi_N\big(rd(\theta)\big) - 2\alpha\varphi_N'\big(rd(\theta)\big) + \varphi_N''\big(rd(\theta)\big)\right]\right.$$

$$\left. + \mathcal{O}(N^{-3/2})\left[-\alpha\varphi_N\big(rd(\theta)\big) + \varphi_N'\big(rd(\theta)\big)\right]\right\},$$

and elementary estimates give, for large N,

$$\left\|Q_\alpha^\Omega u_N - (k^2 - \alpha^2)u_N\right\|_{L^2(\Omega)} = \mathcal{O}(N^{\frac{\nu-2}{2}}).$$

Therefore,

$$\frac{\left\|Q_\alpha^\Omega u_N - (k^2 - \alpha^2)u_N\right\|_{L^2(\Omega)}}{\|u_N\|_{L^2(\Omega)}} \xrightarrow{N\to+\infty} 0,$$

which gives $k^2 - \alpha^2 \in \operatorname{spec} Q_\alpha^\Omega$. As $k > 0$ is arbitrary and $\operatorname{spec} Q_\alpha^\Omega$ is a closed set, Eq. (4) follows. As the set $[-\alpha^2, +\infty)$ has no isolated points, Eq. (4) implies the inclusion $[-\alpha^2, +\infty) \subset \operatorname{spec}_{\mathrm{ess}} Q^\Omega$, and the combination with (3) gives the required equality. □

3. Conical domains with a finite discrete spectrum

We are going to describe first a class of conical domains whose discrete spectrum is (at most) finite. We will start with the following assertion which is essentially an adaptation of the case considered by Exner and Minakov [7, Theorem 5.3] in two dimensions.

Theorem 2. *Let $\nu \geq 2$ and $\Omega \subset \mathbb{R}^\nu$ be an open set with a piecewise C^2 boundary such that the complement $\mathbb{R}^\nu \setminus \Omega$ is convex, then*

$$\operatorname{spec} Q_\alpha^\Omega \subset [-\alpha^2, +\infty) \text{ for any } \alpha > 0.$$

Proof. Let n be the outer unit normal, which is defined at least on the smooth part S_0 of the boundary of Ω. Consider the set $\Theta := \Phi(S_0 \times \mathbb{R}_+)$ with

$$\Phi(s,t) = s - tn(s), \quad s \in S_0, \quad t \in \mathbb{R}_+.$$

Due to the convexity of $\mathbb{R}^\nu \setminus \Omega$ we have the inclusion $\Theta \subset \Omega$, and the map Φ is a bijection between $S_0 \times \mathbb{R}_+$ and Θ. Furthermore, passing to the coordinates $(s,t) \in S_0 \times \mathbb{R}_+$ one obtains, for any $u \in C_c^\infty(\mathbb{R}^\nu)$,

$$\int_\Omega |\nabla u|^2 \mathrm{d}x - \alpha \int_{\partial\Omega} u^2 \mathrm{d}\sigma + \alpha^2 \int_\Omega u^2 \mathrm{d}x$$

$$= \int_\Omega |\nabla u|^2 \mathrm{d}x - \alpha \int_{S_0} u^2 \mathrm{d}\sigma + \alpha^2 \int_\Omega u^2 \mathrm{d}x$$

$$\geq \int_\Theta |\nabla u|^2 \mathrm{d}x - \alpha \int_{S_0} u^2 \mathrm{d}\sigma + \alpha^2 \int_\Theta u^2 \mathrm{d}x$$

$$= \int_{S_0} \int_{\mathbb{R}_+} \left|(\nabla u)\big(\Phi(s,t)\big)\right|^2 J(s,t)\mathrm{d}t\,\mathrm{d}\sigma(s) - \alpha \int_{S_0} u\big(\Phi(s,0)\big)^2 \mathrm{d}\sigma$$

$$+ \alpha^2 \int_{S_0} \int_{\mathbb{R}_+} u\big(\Phi(s,t)\big)^2 J(s,t)\mathrm{d}t\,\mathrm{d}\sigma(s) =: A,$$

where $J(s,t) = \big(1 - tk_1(s)\big) \cdot \ldots \cdot \big(1 - tk_{\nu-1}(s)\big)$ with k_j being the principal curvatures of the boundary $\partial\Omega$. Due to the convexity of $\mathbb{R}^\nu \setminus \Omega$ we have $k_j \leq 0$ and, subsequently, $J \geq 1$, which gives

$$
\begin{aligned}
A &\geq \int_{S_0} \int_{\mathbb{R}_+} \big|(\nabla u)\big(\Phi(s,t)\big)\big|^2 \mathrm{d}t\,\mathrm{d}\sigma(s) - \alpha \int_{S_0} u\big(\Phi(s,0)\big)^2 \mathrm{d}\sigma(s) \\
&\quad + \alpha^2 \int_{S_0} \int_{\mathbb{R}_+} u\big(\Phi(s,t)\big)^2 \mathrm{d}t\,\mathrm{d}\sigma(s) \\
&\geq \int_{S_0} \int_{\mathbb{R}_+} \big|\langle n(s), (\nabla u)\big(\Phi(s,t)\big)\rangle\big|^2 \mathrm{d}t\,\mathrm{d}\sigma(s) - \alpha \int_{S_0} u\big(\Phi(s,0)\big)^2 \mathrm{d}\sigma(s) \\
&\quad + \alpha^2 \int_{S_0} \int_{\mathbb{R}_+} u\big(\Phi(s,t)\big)^2 \mathrm{d}t\,\mathrm{d}\sigma(s) \\
&= \int_{S_0} \int_{\mathbb{R}_+} \Big|\frac{\partial}{\partial t} u\big(\Phi(s,t)\big)\Big|^2 \mathrm{d}t\mathrm{d}\sigma(s) - \alpha \int_{S_0} u\big(\Phi(s,0)\big)^2 \mathrm{d}\sigma \\
&\quad + \alpha^2 \int_{S_0} \int_{\mathbb{R}_+} u\big(\Phi(s,t)\big)^2 \mathrm{d}t\mathrm{d}\sigma(s) \\
&= \int_{S_0} \Big(\int_{\mathbb{R}_+} \Big|\frac{\partial}{\partial t} u\big(\Phi(s,t)\big)\Big|^2 \mathrm{d}t - \alpha u\big(\Phi(s,0)\big)^2 + \alpha^2 \int_{\mathbb{R}_+} u\big(\Phi(s,t)\big)^2 \mathrm{d}t \Big) \mathrm{d}\sigma(s).
\end{aligned}
$$

As for each $v \in H^1(\mathbb{R}_+)$ we have

$$
\int_{\mathbb{R}_+} v'(t)^2 \mathrm{d}t - \alpha v(0)^2 + \alpha^2 \int_{\mathbb{R}_+} v(t)^2 \mathrm{d}t \geq 0,
$$

we arrive at $A \geq 0$ and

$$
\int_\Omega |\nabla u|^2 \mathrm{d}x - \alpha \int_{\partial\Omega} u^2 \mathrm{d}\sigma + \alpha^2 \int_\Omega u^2 \mathrm{d}x \geq 0, \quad u \in C_c^\infty(\mathbb{R}^\nu).
$$

By density, this extends to all $u \in H^1(\Omega)$ and gives $Q_\alpha^\Omega \geq -\alpha^2$. $\qquad\square$

By combining Theorems 1 and 2 we obtain the following result:

Corollary 3. *Let $\nu \geq 2$ and $\Omega \subset \mathbb{R}^\nu$ be a conical domain smooth at infinity and such that $\mathbb{R}^\nu \setminus \Omega$ is convex, then $\mathrm{spec}\, Q_\alpha^\Omega = [-\alpha^2, +\infty)$ for any $\alpha > 0$. In particular, Q_α^Ω has no discrete spectrum for any $\alpha > 0$.*

A slight modification of the above discussion gives a sufficient condition for the finiteness of the discrete spectrum in more general situations.

Theorem 4. *Let $\nu \geq 2$ and $\Omega \subset \mathbb{R}^\nu$ be a conical domain smooth at infinity and $\Lambda = \Lambda(\Omega)$ be the associated cone. Assume that the complement $\mathbb{R}^\nu \setminus \Lambda$ is convex, then the discrete spectrum of Q_α^Ω is finite for any $\alpha > 0$.*

Proof. For $r > 0$, denote $B_r := \{x \in \mathbb{R}^\nu : |x| < r\}$. Assume that the origin is a vertex of Λ and choose $R > 0$ such that Ω coincides with Λ outside B_{R-1}. Denote $\Omega_i := B_R \cap \Omega$

and $\Omega_e := \Omega \setminus \overline{\Omega_i}$ and consider the quadratic form

$$\widetilde{q}(u,u) := \int_{\Omega_i} |\nabla u|^2 \mathrm{d}x - \alpha \int_{\partial\Omega_i \cap \partial\Omega} u^2 \mathrm{d}\sigma$$
$$+ \int_{\Omega_e} |\nabla u|^2 \mathrm{d}x - \alpha \int_{\partial\Omega_e \cap \partial\Omega} u^2 \mathrm{d}\sigma, \quad u \in H^1(\Omega_i \cup \Omega_e),$$

which generates a self-adjoint operator $\widetilde{Q}$ in $L^2(\Omega)$. As $\widetilde{q}$ extends q_α^Ω, due to the min-max principle it is sufficient to show that $\widetilde{Q}$ has finitely many eigenvalues in $(-\infty, -\alpha^2)$. Remark that $\widetilde{Q} = Q_i \oplus Q_e$, where Q_i and Q_e are the self-adjoint operators acting respectively in $L^2(\Omega_i)$ and $L^2(\Omega_e)$ and generated by the quadratic forms

$$q_i(u,u) := \int_{\Omega_i} |\nabla u|^2 \mathrm{d}x - \alpha \int_{\partial\Omega_i \cap \partial\Omega} u^2 \mathrm{d}\sigma, \quad u \in H^1(\Omega_i),$$
$$q_e(u,u) := \int_{\Omega_e} |\nabla u|^2 \mathrm{d}x - \alpha \int_{\partial\Omega_e \cap \partial\Omega} u^2 \mathrm{d}\sigma, \quad u \in H^1(\Omega_e)$$

respectively. Denote $S_e := \partial\Omega_e \cap \partial\Omega \subset \partial\Lambda \cap \partial\Omega$ and consider the map

$$S_e \times \mathbb{R}_+ \ni (s,t) \mapsto \Phi(s,t) := s - tn(s),$$

where n is the outer unit normal. Due to the convexity of $\mathbb{R}^\nu \setminus \Lambda$ the map Φ is injective, and $\Phi(s,t) \in \Lambda$ for any admissible (s,t). On the other hand, as $S_e \subset \partial\Lambda$, we have $s \perp n(s)$ and, hence, $|\Phi(s,t)| \geq |s| > R$, i.e. $\Phi(s,t) \in \mathbb{R}^\nu \setminus \overline{B_R}$. Therefore, $\Phi(S_e \times \mathbb{R}_+) \subset \Lambda \setminus \overline{B_R} \equiv \Omega_e$. Proceeding as in the proof of Theorem 2 we obtain $Q_e \geq -\alpha^2$, which means that the spectrum of $\widetilde{Q}$ in $(-\infty, -\alpha^2)$ coincides with that of Q_i. On the other hand Q_i is semibounded from below and has a compact resolvent and, hence, can have only finitely many eigenvalues in this interval, which gives the sought result. $\square$

4. Conical domains with an infinite discrete spectrum

Now we are going to go in the opposite direction and to show that Robin laplacians on a large class of conical domains have an infinite discrete spectrum.

Lemma 5. *Let $\nu \geq 3$ and $\Lambda \in \mathbb{R}^\nu$ be a smooth cone with a vertex at the origin. Assume that there exists a point on $\partial\Lambda \setminus \{0\}$ at which the mean curvature is strictly positive. Then for any $\alpha > 0$, $r_0 > 0$, $N \in \mathbb{N}$ there exist N functions $F_j \in H^1(\Lambda) \setminus \{0\}$, $j = 1, \ldots, N$, satisfying the following conditions:*
  (a) *$F_j(x) = 0$ for $|x| \leq r_0$ and $j = 1, \ldots, N$,*
  (b) *the functions F_j have mutually disjoint supports,*
  (c) *for any $j = 1, \ldots, N$ there holds*

$$q_\alpha^\Lambda(F_j, F_j) \leq -\alpha^2 \|F_j\|_{L^2(\Lambda)}^2. \tag{5}$$

Proof. We remark first that if the mean curvature H is positive at some point $P \in \partial\Lambda \setminus \{0\}$, then the same holds on the whole half-line $\mathbb{R}_+ P \subset \partial\Lambda$.

For $R > 0$, introduce a unitary transform $U_R : L^2(\Lambda) \to L^2(\Lambda)$ by $(U_R f)(x) = R^{\nu/2} f(Rx)$. It is easy to check the identity $R^2 Q_\alpha^\Lambda = U_R^* Q_{R\alpha}^\Lambda U_R$. In particular, for $F \in H^1(\Lambda)$ we have

$$\frac{q_\alpha^\Lambda(F,F)}{\|F\|_{L^2(\Lambda)}^2} = \frac{1}{R^2} \frac{q_{R\alpha}^\Lambda(U_R F, U_R F)}{\|U_R F\|_{L^2(\Lambda)}^2}. \tag{6}$$

We will look for functions F_j with the above properties using the representation $U_R F_j = f_j$, i.e. $F_j = U_R^* f_j = U_{1/R} f_j$, where R is a large parameter and f_j are new functions to be chosen in a suitable way.

For $r > 0$, denote $B_r := \{x \in \mathbb{R}^\nu : |x| < r\}$. Let $b > 2$. By smoothening the vertex of Λ one can construct a connected C^2 smooth domain Θ coinciding with Λ outside the ball B_{b-1}. Clearly, for any function $f \in H^1(\Lambda)$ vanishing in $\Lambda \cap B_b$ we have

$$\frac{q_{R\alpha}^\Lambda(f, f)}{\|f\|_{L^2(\Lambda)}^2} = \frac{q_{R\alpha}^\Theta(\widetilde{f}, \widetilde{f})}{\|\widetilde{f}\|_{L^2(\Theta)}^2}, \tag{7}$$

where $\widetilde{f} = f$ on $\Theta \setminus B_{b-1}$ and $\widetilde{f} = 0$ on the remaining part of Θ. Now we construct a function $\widetilde{f}$ with the above property in a special way using some computations of [20].

Denote $S := \partial\Theta$ and let $S \ni s \mapsto n(s)$ be the outer unit normal. Furthermore, let $\delta := \delta(R)$ be a positive function such that $\delta(R) \to 0$ and $R\delta(R) \to +\infty$ as $R \to +\infty$. Denote by E and ψ respectively the first eigenvalue and the associated normalized eigenfunction of the operator $\phi \mapsto -\phi''$ in $L^2(0, \delta)$ with the boundary conditions $\phi'(0) + R\alpha\phi(0) = 0$ and $\phi(\delta) = 0$. We recall that one has the estimate $-\alpha^2 R^2 \leq E \leq -\alpha^2 R^2 + \mathcal{O}(\alpha^2 R^2 e^{-\delta R\alpha})$ as $R \to +\infty$, see [20, Lemma 2.1].

Consider the map $S \times (0, \delta) \ni (s, t) \mapsto \Phi(s, t) := s - t n(s) \in \mathbb{R}^\nu$. If R is sufficiently large, then Φ is a diffeomorphism between $S \times (0, \delta)$ and $\Phi\big(S \times (0, \delta)\big)$ and, moreover, $\Phi\big(S \times (0, \delta)\big) \subset \Theta$. For $v \in H^1(S)$, consider the associated functions $\widetilde{f} = \widetilde{f}_v \in H^1(\Theta)$ given by

$$\widetilde{f}(x) \equiv \widetilde{f}_v(x) := \begin{cases} v(s)\psi(t), & x = \Phi(s, t), \quad (s, t) \in S \times (0, \delta), \\ 0, & x \in \Theta \setminus \Phi\big(S \times (0, \delta)\big). \end{cases} \tag{8}$$

By [20, Lemma 4.1], one can find constants c_1', c_2', c_3' and R_0 such that for all $v \in H^1(S)$ and $R \geq R_0$ there holds

$$\frac{q_{R\alpha}^\Theta(\widetilde{f}, \widetilde{f})}{\|\widetilde{f}\|_{L^2(\Theta)}^2} - E$$

$$\leq (1 + c_1'\delta) \frac{(1 + c_2'\delta) \displaystyle\int_S g^{\rho\mu} \partial_\rho v\, \partial_\mu v\, \mathrm{d}\sigma - R\alpha(\nu - 1) \displaystyle\int_S H v^2\, \mathrm{d}\sigma}{\|v\|_{L^2(S, \mathrm{d}\sigma)}^2}$$

$$+ c_3'(1 + R\alpha e^{-R\alpha\delta}),$$

where $(g^{\rho\mu})$ is the contravariant metric tensor on S induced by the embedding into $\mathbb{R}^\nu$ and Einstein convention for indices is used. In order to simplify the subsequent computations we take $\delta := R^{-1/2}$, then there are positive constants b_1, b_2, b_3 and R_1 such that for all $v \in H^1(S)$ and $R \geq R_1$ there holds

$$\frac{q_{R\alpha}^\Theta(\widetilde{f}, \widetilde{f})}{\|\widetilde{f}\|_{L^2(\Theta)}^2} \leq -\alpha^2 R^2 + \frac{b_1 \displaystyle\int_S g^{\rho\mu} \partial_\rho v\, \partial_\mu v\, \mathrm{d}\sigma - b_2\alpha R \displaystyle\int_S H v^2 \mathrm{d}\sigma}{\|v\|_{L^2(S, \mathrm{d}\sigma)}^2} + b_3. \tag{9}$$

Choose a point $M \in \partial\Sigma_\Lambda$ with $H(M) = \max_{s \in \partial\Sigma_\Lambda} H(s)$. By assumption, $H(M) > 0$. By applying a suitable rotation we may assume that M has the coordinates $(1, 0, \ldots, 0)$ and that $n(M) = (0, \ldots, 0, -1)$. It follows that the hyperplane $x_\nu = 0$ is tangent to Λ at M,

and there exists a C^2 function h with $h(0,\ldots,0) = 0$ and $\nabla h(0,\ldots,0) = 0$ and some $\varepsilon > 0$ and $C > 0$ such that

$$\Lambda \cap \{x_1 = 1\} \cap \bigcap_{k=2}^{\nu-1} \{|x_k| < \varepsilon\} \cap \{|x_\nu| < C\varepsilon\}$$

$$= \{x_\nu > h(x_2,\ldots,x_{\nu-1})\} \cap \{x_1 = 1\} \cap \bigcap_{k=2}^{\nu-1} \{|x_k| < \varepsilon\} \cap \{|x_\nu| < C\varepsilon\}.$$

It follows that

$$\partial\Lambda \cap \{x_1 > 1\} \cap \bigcap_{k=2}^{\nu-1} \{|x_k| < \varepsilon x_1\} \cap \{|x_\nu| < C\varepsilon x_1\}$$

$$= \Big\{ (x_1,\ldots,x_\nu) : x_1 > 1, \ |x_k| < \varepsilon x_1, \ k = 2,\ldots,\nu-1,$$

$$x_\nu = x_1 h\Big(\frac{x_2}{x_1},\ldots,\frac{x_{\nu-1}}{x_1}\Big) \Big\}.$$

As S coincides with $\partial\Lambda$ for $|x| > b$, one can use $(x_1,\ldots,x_{\nu-1})$ as local coordinates on a part $S_{b,\varepsilon}$ of S defined as $S_{b,\varepsilon} := X(\Pi_{b,\varepsilon})$, where

$$\Pi_{b,\varepsilon} := \big\{ (x_1,\ldots,x_{\nu-1}) : x_1 \geq b, \quad |x_k| < \varepsilon x_1, \quad k = 2,\ldots,\nu-1 \big\},$$

$$X(x_1,\ldots,x_{\nu-1}) := \begin{pmatrix} x_1 \\ \cdots \\ x_{\nu-1} \\ x_1 h\big(\frac{x_2}{x_1},\ldots,\frac{x_{\nu-1}}{x_1}\big) \end{pmatrix}.$$

In particular, for the associated metric tensor g_{jk} we have

$$g_{jk}(x_1,\ldots,x_{\nu-1}) = \frac{\partial X}{\partial x_j} \cdot \frac{\partial X}{\partial x_k} = \delta_{jk} + a_{jk},$$

where δ_{jk} is Kronecker delta,

$$a_{11} = \Big[h\Big(\frac{x_2}{x_1},\ldots,\frac{x_{\nu-1}}{x_1}\Big) - \sum_{\ell=2}^{\nu-1} \frac{x_\ell}{x_1} h'_{\ell-1}\Big(\frac{x_2}{x_1},\ldots,\frac{x_{\nu-1}}{x_1}\Big) \Big]^2,$$

$$a_{1j} = a_{j1} = \Big[h\Big(\frac{x_2}{x_1},\ldots,\frac{x_{\nu-1}}{x_1}\Big) - \sum_{\ell=2}^{\nu-1} \frac{x_\ell}{x_1} h'_{\ell-1}\Big(\frac{x_2}{x_1},\ldots,\frac{x_{\nu-1}}{x_1}\Big) \Big]$$

$$\times h'_{j-1}\Big(\frac{x_2}{x_1},\ldots,\frac{x_{\nu-1}}{x_1}\Big), \quad j = 2,\ldots,\nu-1,$$

$$a_{jk} = h'_{j-1}\Big(\frac{x_2}{x_1},\ldots,\frac{x_{\nu-1}}{x_1}\Big) h'_{k-1}\Big(\frac{x_2}{x_1},\ldots,\frac{x_{\nu-1}}{x_1}\Big), \quad j,k = 2,\ldots,\nu-1,$$

and h'_ℓ stands for the partial derivative of h with respect to the ℓth variable. In particular, one may assume that the parameter ε in the above constructions is sufficiently small to

have $\left\|(a_{jk})\right\| \leq \frac{1}{2}$, which then implies, for $x \in \Pi_{b,\varepsilon}$,

$$\frac{1}{2} \leq (g_{jk}) \leq \frac{3}{2}, \quad (g^{jk}) := (g_{jk})^{-1} \leq 2, \tag{10}$$

$$\left(\frac{1}{2}\right)^{(\nu-1)/2} \leq \sqrt{\det g} \leq \left(\frac{3}{2}\right)^{(\nu-1)/2}. \tag{11}$$

In addition, let us pick $A \in \left(0, H(M)\right)$, then due to the continuity of H we may assume that ε is sufficiently small to have

$$H\big(X(1, x_2, \ldots, x_{\nu-1})\big) \geq A \text{ for } |x_k| \leq \varepsilon, \quad k = 2, \ldots, \nu-1,$$

and, hence,

$$H\big(X(x_1, x_2, \ldots, x_{\nu-1})\big) \geq \frac{A}{x_1} \text{ for } x \in \Pi_{b,\varepsilon}.$$

For $\varphi \in C_c^\infty(\Pi_{b,\varepsilon})$ define $v \equiv v_\varphi \in H^1(S)$ through

$$v\big(X(x_1, x_2, \ldots, x_{\nu-1})\big) := \varphi(x_1, \ldots, x_{\nu-1})$$

and extend it by zero to the whole of S. Using the representation $d\sigma = \sqrt{\det g}\, dx_1 \ldots dx_{\nu-1}$ and the estimates (10) and (11) we obtain, as $R \geq R_1$,

$$\frac{q_{R\alpha}^\Theta(\widetilde{f}, \widetilde{f})}{\|\widetilde{f}\|_{L^2(\Theta)}^2} \leq -\alpha^2 R^2$$

$$+ \frac{c_1 \int_{\Pi_{b,\varepsilon}} |\nabla\varphi|^2 dx_1 \ldots dx_{\nu-1} - c_2 \alpha R \int_{\Pi_{b,\varepsilon}} \dfrac{\varphi^2}{x_1} dx_1 \ldots dx_{\nu-1}}{\displaystyle\int_{\Pi_{b,\varepsilon}} \varphi^2 dx_1 \ldots dx_{\nu-1}} + c_3$$

with $c_1 := 2 \cdot 3^{(\nu-1)/2} b_1 > 0$, $c_2 := A \cdot 3^{(1-\nu)/2} b_2 > 0$ and $c_3 := b_3 > 0$. Now let us pick $\phi_j \in C_c^\infty(\mathbb{R}^{\nu-1})$ with $\operatorname{supp} \phi_j \subset \Pi_{b,\varepsilon}$, $j = 1, \ldots, N$, not identically zero and with mutually disjoint supports and set

$$\varphi_{j,R}(x) := \phi_j\Big(\frac{x}{\sqrt{R}}\Big),$$

$$m := \sup\Big\{x_1 : \text{ there exists } (x_1, \ldots x_\nu) \in \bigcup_{j=1}^N \operatorname{supp} \phi_j\Big\}.$$

For $R \geq 1$ we still have $\varphi_{j,R} \in C_c^\infty(\Pi_{b,\varepsilon})$ and these new functions still have mutually disjoint supports. Furthermore, set $v_{j,R} := v_{\varphi_{j,R}}$ and $\widetilde{f}_{j,R} := \widetilde{f}_{v_{j,R}}$, $j = 1, \ldots, N$, see (8), then, by construction, the functions $\widetilde{f}_{j,R}$ are not identically zero and have mutually disjoint supports, and

$$\frac{q_{R\alpha}^\Theta(\widetilde{f}_{j,R}, \widetilde{f}_{j,R})}{\|\widetilde{f}_{j,R}\|_{L^2(\Theta)}^2} \leq -\alpha^2 R^2$$

$$+ \frac{\dfrac{c_1}{R} \int_{\mathbb{R}^{\nu-1}} |\nabla\phi_j|^2 dx_1 \ldots dx_{\nu-1} - \dfrac{c_2 \alpha \sqrt{R}}{m} \int_{\mathbb{R}^{\nu-1}} \phi_j^2 dx_1 \ldots dx_{\nu-1}}{\displaystyle\int_{\mathbb{R}^{\nu-1}} \phi_j^2 dx_1 \ldots dx_{\nu-1}} + c_3.$$

In particular, for sufficiently large R we have the strict inequalities

$$\frac{q_{R\alpha}^{\Theta}(\widetilde{f}_{j,R}, \widetilde{f}_{j,R})}{\|\widetilde{f}_{j,R}\|_{L^2(\Theta)}^2} < -\alpha^2 R^2.$$

Remark that by construction for sufficiently large R one also has $\widetilde{f}_{j,R}(x) = 0$ for $|x| \leq b+1$. In particular, one can define $f_{j,R} \in H^1(\Lambda)$ by $f_{j,R} = \widetilde{f}_{j,R}$ on $\Lambda \cap \{|x| \geq b\}$ and $f_{j,R} = 0$ on $\Lambda \cap \{|x| \leq b\}$, then by (7) we have, as R is large,

$$\frac{q_{R\alpha}^{\Lambda}(f_{j,R}, f_{j,R})}{\|f_{j,R}\|_{L^2(\Lambda)}^2} < -\alpha^2 R^2.$$

Finally, define $F_j(\cdot) \equiv F_{j,R}(\cdot) := R^{\nu/2} f_{j,R}(\cdot/R)$, then (5) holds due to (6). By construction, F_j are not identically zero, have mutually disjoint supports and vanish in any prescribed ball if R is large. Hence, the family (F_j) satisfies all the requested conditions. $\qquad\square$

Theorem 6. *Let $\nu \geq 3$ and $\Omega \subset \mathbb{R}^\nu$ be a conical domain smooth at infinity. Assume that outside any ball there is a point at the boundary of Ω at which the mean curvature is strictly positive, then for any $\alpha > 0$ the discrete spectrum of Q_α^Ω is infinite.*

Proof. Consider the cone $\Lambda = \Lambda(\Omega)$. Without loss of generality one may assume that the origin is a vertex of Λ. Let $r_0 > 0$ be such that Λ and Ω coincide in $\{x \in \mathbb{R}^\nu : |x| > r_0 - 1\}$. Take any $N \in \mathbb{N}$ and construct functions $F_1, \dots, F_N \in H^1(\Lambda)$ as in Lemma 5. The subspace L spanned by the functions F_j is then N-dimensional. For $F \in L$, define $\widetilde{F} \in H^1(\Omega)$ by $\widetilde{F} = F$ on $\Lambda \cap \Omega$ and $\widetilde{F} = 0$ on the remaining part of Ω. We have then $q_\alpha^\Omega(\widetilde{F}, \widetilde{F}) = q_\alpha^\Lambda(F, F)$ and $\|\widetilde{F}\|_{L^2(\Omega)} = \|F\|_{L^2(\Lambda)}$ for any $F \in L$, and the subspace $\widetilde{L} := \{\widetilde{F} : F \in L\} \subset H^1(\Omega)$ is N-dimensional. Due to the choice of F_j we have

$$E_N(Q_\alpha^\Omega) \leq \sup_{G \in \widetilde{L}, G \not\equiv 0} \frac{q_\alpha^\Omega(G, G)}{\|G\|_{L^2(\Omega)}^2} = \sup_{F \in L, F \not\equiv 0} \frac{q_\alpha^\Omega(\widetilde{F}, \widetilde{F})}{\|\widetilde{F}\|_{L^2(\Omega)}^2}$$

$$= \sup_{F \in L, F \not\equiv 0} \frac{q_\alpha^\Lambda(F, F)}{\|F\|_{L^2(\Lambda)}^2} < -\alpha^2 = \inf \mathrm{spec}_{\mathrm{ess}} Q_\alpha^\Omega.$$

Hence, Q_α^Ω has at least N eigenvalues in $(-\infty, -\alpha^2)$ by the min-max principle. As N is arbitrary, the result follows. $\qquad\square$

5. Conical domains in three dimensions

It appears that the main assumption of Theorem 6 (the positivity of the mean curvature on an unbounded set) is quite simple to check for three-dimensional cones, as the following assertion shows.

Lemma 7. *Let Λ be a cone in $\mathbb{R}^3$ with a simply connected smooth cross section $\Sigma := \Sigma_\Lambda \subset \mathbb{S}^2$. Assume that the set $\mathbb{R}^\nu \setminus \Lambda$ is* not *convex, then there is half-line on the boundary of $\partial\Lambda$ on which the mean curvature is strictly positive.*

Proof. We may assume that the origin is a vertex of Λ. By assumption, the boundary $\partial\Sigma$ is a simple smooth curve on $\mathbb{S}^2$. Let ℓ be its length and $\gamma : \mathbb{R}/\ell\mathbb{Z} \to \mathbb{S}^2 \subset \mathbb{R}^3$ be its

arc-length parametrization chosen such that the vector $\gamma \times \gamma'$ points to the outside of Λ. It is elementary to check that the map

$$(\mathbb{R}/\ell\mathbb{Z}) \times \mathbb{R}_+ \ni (s,t) \mapsto X(s,t) := t\gamma(s)$$

gives a parametrization of the boundary of Λ and that the mean curvature $H(s,t)$ at the point $X(s,t)$ is given by $H(s,t) = \kappa(s)/(2t)$, where $\kappa(s) = \det\big(\gamma(s), \gamma'(s), \gamma''(s)\big)$ is the geodesic curvature of $\partial\Sigma$ at the point $\gamma(s)$. If $\kappa \leq 0$ at all points, then the complement $\mathbb{S}^2 \setminus \Sigma$ is geodesically convex, see e.g. [3, Proposition 2.1] and then the complement $\mathbb{R}^3 \setminus \Lambda$ in its turn should be convex too, which contradicts the assumption. Therefore, there exists s_0 such that $\kappa(s_0) > 0$, and then $H(s_0, t) > 0$ for all $t \in \mathbb{R}_+$. $\qquad\square$

The combination of Theorem 6 with Lemma 7 gives the following result:

Corollary 8. *Let $\Omega \subset \mathbb{R}^3$ be a conical domain smooth at infinity. Assume that the associated cone $\Lambda(\Omega)$ has a simply connected cross section and that the complement $\mathbb{R}^3 \setminus \Lambda(\Omega)$ is not a convex set, then Q_α^Ω has an infinite discrete spectrum for any $\alpha > 0$.*

We point out the following fact concerning the three-dimensional smooth cones:

Corollary 9. *Let $\Lambda \subset \mathbb{R}^3$ be a cone with a simply connected smooth cross section and $\alpha > 0$, then the discrete spectrum of Q_α^Λ is either infinite or empty.*

Proof. If $\mathbb{R}^3 \setminus \Lambda$ is convex, then the discrete spectrum is empty by Corollary 3, otherwise it is infinite by Corollary 8. $\qquad\square$

Finally, the combination of Theorem 4 with Corollary 8 gives the following result:

Corollary 10. *Let $\alpha > 0$ and $\Omega \subset \mathbb{R}^3$ be a conical domain such that the associated cone $\Lambda(\Omega)$ has a simply connected smooth cross section, then the discrete spectrum of Q_α^Ω is finite if and only if the complement $\mathbb{R}^\nu \setminus \Lambda(\Omega)$ is a convex set.*

Remark 11. We remark that there is no analog of Corollaries 8 and 9 for non-smooth cones. As an example one can consider Q_α^Ω for the convex cone $\Omega := (\mathbb{R}_+)^\nu \subset \mathbb{R}^\nu$. The separation of variables gives, for any $\alpha > 0$,

$$\mathrm{spec}_{\mathrm{ess}}\, Q_\alpha^\Omega = \big[-(\nu - 1)\alpha^2, +\infty\big),$$
$$\mathrm{spec}_{\mathrm{disc}}\, Q_\alpha^\Omega = \{-\nu\alpha^2\}, \quad \dim\ker(Q_\alpha^\Omega + \nu\alpha^2) = 1.$$

The example also shows that Corollary 10 cannot be extended to the two-dimensional case.

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